MMATH FINAL EXAMINATION ALGEBRAIC GEOMETRY

Attempt all questions. Assume that the base field k, in all questions below, is algebraically closed. Total: 50 marks. Time: 3 hours.

(1) Prove that up to projective equivalence, there is only one irreducible projective plane cubic curve with a node: $F(x,y,z) = x^3 + y^3 - xyz$. (8 marks)

(2) Let W be the set of all conics in $\mathbb{P}^2$. Show that W can be identified with $\mathbb{P}^5$. Let P_1, P_2, P_3, P_4 be four distinct points of $\mathbb{P}^2$, and let V be the subset of all conics in $\mathbb{P}^2$ passing through these four points. Show that V is a linear subspace (defined by homogenous linear equations) of $W(= \mathbb{P}^5)$. Show that $\dim(V) = 2$ if these four points lie on a line, and $\dim(V) = 1$ otherwise. (8 marks)

(3) Let C_1, C_2 be two irreducible projective curves, let f be a non-constant rational map from C_1 to C_2. Prove that f is dominating (or dominant), and $k(C_1)$ is a finite field extension of $f^*(k(C_2))$. Here $f^* : k(C_2) \to k(C_1)$ is the induced map on function fields. (8 marks)

(4) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birationally equivalent but not isomorphic. (8 marks)

(5) Let C be an irreducible projective plane curve and let $P_1, \ldots, P_n$ be simple points on C. Let $m_1, \ldots, m_n$ be arbitrary integers. Show that there exists a rational function $z \in k(C)$ such that $\mathrm{ord}_{P_i}(z) = m_i$ for $i = 1, \ldots, n$ (note there are no conditions on the points other than $P_1, \ldots, P_n$, that is, z may have other zeroes). (9 marks)

(6) Let $C \subset \mathbb{P}^2_k$ be a nonsingular projective plane cubic curve. Let L be a line in $\mathbb{P}^2_k$ which intersects C in three distinct points. Show that if two of these points are flexes on C, then so is the third point. (9 marks)